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Case of Updating the Factorized Covariance Matrix

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I. Introduction

IN some applications, such as the Gunship AC-130U navigation system, a special type of measurement update is required from time to time, the so-called manual updates. This type of update amounts to implementing an imposed Kalman filter gain at a specified time.

The main objective is to perform the update of the covariance matrix in $U-D$ form without reconstructing and decomposing the covariance matrix.

As is well known, the covariance matrix updating process can be described by Joseph's variant (Kalman stabilized)

$$P^+ = (I-KH)P^-(I-KH)^T + KRK^T \tag{1}$$

where P^-, P^+ is the covariance matrix prior and after measurement incorporation of $n\times n$, respectively, K the Kalman gain matrix of $n\times m$, H the observation matrix of $m\times n$, and R the measurement variance matrix of $m\times m$.

When a specific value is imposed on the Kalman gain matrix K (also called the suboptimal gain), we can determine a special structure of the updated covariance matrix P^+. For example, if we consider $m=2$, $n>2$, and

$$K = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0_{n-2,2}} \end{bmatrix}, \qquad H = \left[\begin{array}{cc|c} 1 & 0 & \\ 0 & 1 & 0_{2,n-2} \end{array}\right]$$

then

$$P^+ = \begin{bmatrix} R & 0_{2,n-2} \\ 0_{2,n-2} & P^-(n-2,n-2) \end{bmatrix}$$

where $P^-(n-2,n-2)$ is the $(n-2)\times(n-2)$ lower-right block of the matrix P^-, and 0_{ij} the null matrix with i rows and j columns.

In general, the structure of the updated covariance matrix after implementing one update can be described as follows:

$$P^+ = \begin{bmatrix} P_{11}^- & 0_{p_1p_2} & P_{13}^- \\ 0_{p_2p_1} & P_{\text{imp}} & 0_{p_2p_3} \\ P_{31}^- & 0_{p_3p_2} & P_{33}^- \end{bmatrix} \tag{2}$$

where P_{11}^-, P_{13}^-, P_{31}^-, and P_{33}^- are submatrices from the covariance matrix P^- of appropriate dimensions, i.e., P^+ is obtained from P^- by substituting p_2 rows and p_2 columns as specified in Eq. (2). The matrix P_{imp}, of $p_2\times p_2$, is symmetric positive definite.

II. Presentation of the Method

Next, we focus our attention on determining the (U^+,D^+) factors of the updated covariance matrix P^+ as a function of the (U^-,D^-) factors of the covariance matrix P^- and the $(U_{\text{imp}},D_{\text{imp}})$ factors of the imposed submatrix P_{imp} when the structure [Eq. (2)] is assumed.

Let us consider a general covariance matrix P with the same partitions as specified in Eq. (2) and the corresponding (U,D) factors using the same type of partitions; then we have

$$P = UDU^T = \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix}^T$$

$$= \begin{bmatrix} U_{11}D_1U_{11}^T + U_{12}D_2U_{12}^T + U_{13}D_3U_{13}^T & U_{12}D_2U_{22}^T + U_{13}D_3U_{23}^T & U_{13}D_3U_{33}^T \\ \text{symmetric} & U_{22}D_2U_{22}^T + U_{23}D_3U_{23}^T & U_{23}D_3U_{33}^T \\ \text{symmetric} & \text{symmetric} & U_{33}D_3U_{33}^T \end{bmatrix} \tag{3}$$

Substituting the $U-D$ structure specified in Eq. (3) for both P^+ and P^- in Eq. (2), and matching the corresponding blocks (in fact only the six blocks contained in the upper triangular part of the covariance matrix) we obtain the following equations:

$$U_{11}^+D_1^+U_{11}^{+T} + U_{12}^+D_2^+U_{12}^{+T} + U_{13}^+D_3^+U_{13}^{+T} = U_{11}^-D_1^-U_{11}^{-T} + U_{12}^-D_2^-U_{12}^{-T} + U_{13}^-D_3^-U_{13}^{-T} \tag{4}$$

$$U_{12}^+D_2^+U_{22}^{+T} + U_{13}^+D_3^+U_{23}^{+T} = 0 \tag{5}$$

$$U_{22}^+D_2^+U_{22}^{+T} + U_{23}^+D_3^+U_{23}^{+T} = P_{\text{imp}} = U_{\text{imp}}D_{\text{imp}}U_{\text{imp}}^T \tag{6}$$

$$U_{13}^+D_3^+U_{33}^{+T} = U_{13}^-D_3^-U_{33}^{-T} \tag{7}$$

$$U_{23}^+D_3^+U_{33}^{+T} = 0 \tag{8}$$

$$U_{33}^+D_3^+U_{33}^{+T} = U_{33}^-D_3^-U_{33}^{-T} \tag{9}$$

From Eqs. (9), (8), and (7), respectively, it follows that we can set

$$U_{33}^+ = U_{33}^-, \qquad D_3^+ = D_3^- \tag{10}$$

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$$U_{23}^{+} = 0 \tag{11}$$

$$U_{13}^{+} = U_{13}^{-} \tag{12}$$

Substituting Eq. (11) in Eq. (6) we obtain

$$U_{22}^{+} D_{2}^{+} U_{22}^{+T} = U_{\text{imp}} D_{\text{imp}} U_{\text{imp}}^{T} \tag{13}$$

and we can set

$$U_{22}^{+} = U_{\text{imp}}, \qquad D_{2}^{+} = D_{\text{imp}} \tag{14}$$

Substituting Eq. (11) in Eq. (5) we obtain

$$U_{12}^{+} D_{2}^{+} U_{22}^{+T} = 0 \tag{15}$$

and we can select

$$U_{12}^{+} = 0 \tag{16}$$

Taking into account Eqs. (10) and (15), Eq. (4) becomes

$$U_{11}^{+} D_{1}^{+} U_{11}^{+T} = U_{11}^{-} D_{1}^{-} U_{11}^{-T} + U_{12}^{-} D_{2}^{-} U_{12}^{-T} \tag{17}$$

We have determined now all of the elements of the updated covariance matrix in $U-D$ form, except for the pair (U_{11}^{+}, D_{1}^{+}). It will be noted that the matrix $U_{12}^{-} D_{2}^{-} U_{12}^{-T}$ can be written as a sum of terms $d_k u_k u_k^T$, where d_k is the kth diagonal element of D_2^- and u_k is the kth column vector of U_{12}^-. To determine the pair (U_{11}^{+}, D_{1}^{+}) from Eq. (17), we can use repeatedly a rank-one factorization algorithm (Agee-Turner algorithm, see Refs. 1 and 2) to update the initial pair (U_{11}^{-}, D_{1}^{-}) with the factors $d_k u_k u_k^T$ of the matrix $U_{12}^{-} D_{2}^{-} U_{12}^{-T}$.

Several remarks can be made regarding the preceding method.

1) If the imposed structure Eq. (2) of the updated covariance matrix does not have the rows and columns related to the submatrix P_{11}^-, i.e., the submatrix P_{imp} is the upper-left block, then the method does not need to use the rank-one factorization algorithm. In this case only the elements

$$U_{33}^{+} = U_{33}^{-}, \qquad D_{3}^{+} = D_{3}^{-}, \qquad U_{23}^{+} = 0$$
$$U_{22}^{+} = U_{\text{imp}}, \qquad D_{2}^{+} = D_{\text{imp}} \tag{18}$$

are used, and the computation is reduced to simple identifications.

2) If the given matrix P_{imp} is not factorized in $U-D$ factors, then the so-called $U-D$ decomposition algorithm (modified Cholesky algorithm[1]) can be used to determine the corresponding pair $(U_{\text{imp}}, D_{\text{imp}})$.

3) Based on the observation made earlier (see remark 1), it is recommended that the selection of the order of the states of the Kalman filter be made keeping in mind that the implementation of the presented method is substantially simplified if the states subject to special updates are the first ones in the state vector.

4) The presented method can also be useful in the process of reinitialization of the covariance matrix after transition from one structure to another when a discrete Kalman filter of $U-D$ form is implemented.

III. Example

Let us consider the following structure for the 6×6 updated covariance matrix:

$$P^{+} = \begin{bmatrix} P_{11}^- & P_{12}^- & 0 & 0 & P_{15}^- & P_{16}^- \\ P_{21}^- & P_{22}^- & 0 & 0 & P_{25}^- & P_{26}^- \\ 0 & 0 & k_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & k_2 & 0 & 0 \\ P_{51}^- & P_{52}^- & 0 & 0 & P_{55}^- & P_{56}^- \\ P_{61}^- & P_{62}^- & 0 & 0 & P_{65}^- & P_{66}^- \end{bmatrix} \tag{19}$$

and we assume that the corresponding (U,D) factors of the covariance matrix P^- are known.

By using Eqs. (10–12), (14), (16), and (17) it results that

$$U^{+} = \begin{bmatrix} 1 & u_{12}^+ & 0 & 0 & u_{15}^+ & u_{16}^+ \\ 0 & 1 & 0 & 0 & u_{25}^+ & u_{26}^+ \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & u_{56}^+ \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{20}$$

$$D^{+} = \text{diag}(d_1^+,\ d_2^+,\ d_3^+,\ d_4^+,\ d_5^+,\ d_6^+) \tag{21}$$

where

$$u_{15}^+ = u_{15}^-, \quad u_{16}^+ = u_{16}^-, \quad u_{25}^+ = u_{25}^-, \quad u_{26}^+ = u_{26}^-, \quad u_{56}^+ = u_{56}^-$$
$$d_3^+ = k_1, \quad d_4^+ = k_2, \quad d_5^+ = d_5^-, \quad d_6^+ = d_6^- \tag{22}$$

and

$$d_2^+ = d_2^- + d_3^- (u_{23}^-)^2 + d_4^- (u_{24}^-)^2 \tag{23}$$

$$u_{12}^+ = (d_2^- u_{12}^- + d_3^- u_{13}^- u_{23}^- + d_4^- u_{14}^- u_{24}^-)/d_2^+ \tag{24}$$

$$d_1^+ = d_1^- + d_2^- (u_{12}^-)^2 + d_3^- (u_{13}^-)^2 + d_4^- (u_{14}^-)^2 - d_2^+ (u_{12}^+)^2 \tag{25}$$

The elements d_1^+, d_2^+, and u_{12}^+ have been determined by using a direct identification method of the elements involved in Eq. (17).

IV. Concluding Remarks

In this Note a new method has been introduced involving the updating of the $U-D$ covariance factors when a special structured updated covariance matrix is required. The computation is executed by using only prior and post-updated $U-D$ covariance factors. The algorithm has been successfully tested and implemented in Gunship navigation real-time software for position and velocity updates.

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